REGULARIZATION IN CELESTIAL MECHANICS

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Summary

Regularization in the context of celestial mechanics or, more generally speaking, in the context of dynamical astronomy means introducing appropriate time and space variables such that the equations of motion of point masses (mostly under Newtonian forces) are regular in binary collisions. This allows insight into the behavior of orbits near collisions as well as efficient computation of collision and near-collision orbits and adequate treatment of binary stars in N-body simulations. In this chapter we begin with Levi-Civita's (1920) regularization of the perturbed planar two-body problem, supplemented by an alternate approach to the theory of Kepler motion. We then proceed to the remarkable extension to three dimensions by Kustaanheimo-Stiefel (1964/1965), using an elegant quaternion formalism. By taking advantage of the Hamiltonian equations of motion, we finally describe global regularization of the spatial restricted three-body problem as well as of the general planar three-body problem.

1. Introduction

Modern dynamical astronomy is based on two fundamental discoveries of the 17th century: the discovery of the laws of planetary motion by Johannes Kepler in 1609 and 1619, and the discovery of the laws of gravitation and the laws of motion under external forces by Isaac Newton in 1687. Clearly, Newton must have been inspired by Kepler's laws; on the other hand, Kepler's laws turn out to be a mathematical consequence of Newton's equations of motion.

Newton's basic laws are formulated by using the abstract notion of the *point mass*, an ideal particle of positive mass $m > 0$ having infinitely small physical size. Although the motion of bodies of finite size is a highly relevant issue in aerospace engineering and in collisional stellar dynamics (see, e.g. *The gravitational two-body problem*), a large part of dynamical astronomy deals with the motion of point masses. In this chapter the central issues are *collisions* of point masses, i.e. events where two point masses approach to arbitrarily small distance: the motion at the instance of collision will have a *singularity*. We will discuss methods of removing singularities from the equations of motion by introducing new coordinates. This procedure is called *regularization*.

Beginning with the fundamentals, we state Newton's well-known law of gravitation which expresses the force $\mathbf{f}$ exerted by the point mass m_1 onto m_2 as a vector of magnitude

$$|\mathbf{f}| = \gamma \frac{m_1 m_2}{r^2} \tag{1}$$

pointing from m_2 to m_1, where γ is the universal constant of gravitation and r is the distance between the two point masses. Throughout the symbol m_j will denote the mass of the j th body as well as the body itself. We will follow the customary notation of celestial mechanics: instead of the masses m_j the *gravitational parameters* $\mu_j = \gamma m_j$ will appear in the equations. The occasional use of the symbol m_j for the gravitational parameter corresponds to using physical units that imply $\gamma = 1$, e.g. the mass of the central body of a reference two-body problem (e.g. the Earth) as the *unit mass*, the radius of the circular reference orbit of a massless particle (e.g. a small satellite) as the *unit length*, and the time of revolution of the satellite, divided by 2π, as the *time unit*.

To get an idea about possible behaviors at collision, we consider the 1-dimensional motion of a massless particle under the influence of the point mass m situated at the origin. The distance $r(t)$ of the particle from the origin, as a function of time t, satisfies the Newtonian differential equation

$$\ddot{r} = -m r^{-2}, \qquad \dot{(\;)} = \frac{d}{dt} \tag{2}$$

which admits a solution of the form $r = ct^{\alpha}$. The exponent α must have the value $\alpha = \frac{2}{3}$, and the constant c is uniquely determined; Eq. (2) has the solution

$$r(t) = ct^{2/3}, \qquad c = \left(\frac{9}{2} m\right)^{1/3}. \tag{3}$$

This motion is referred to as 1-dimensional (or rectilinear) parabolic Kepler motion. $r(t)$ exists for every $t \in \mathbb{R}$; r tends to $+\infty$ for $t \to \pm\infty$, and the collision singularity at

$t=0$ is of the algebraic type $r=O\left(t^{2/3}\right)$.

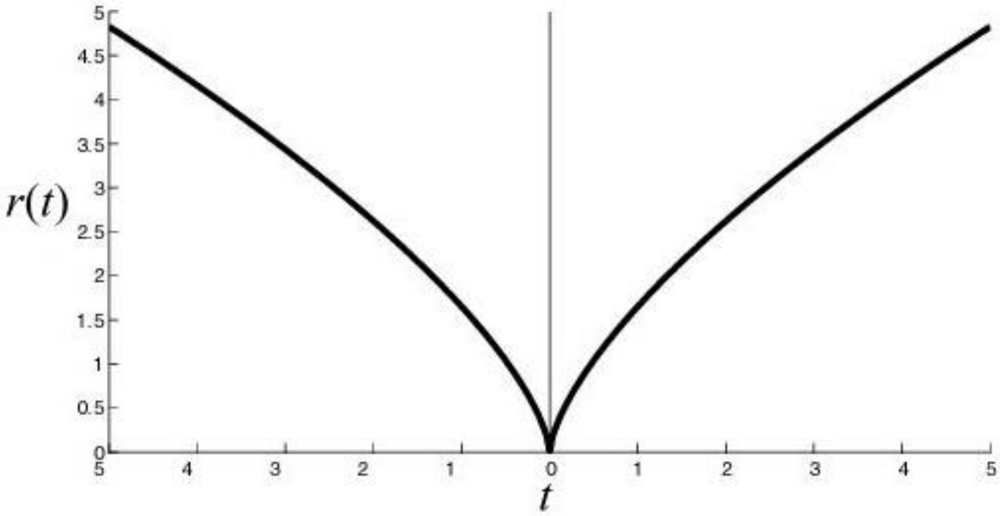

Figure 1. The distance r as a function of time t in rectilinear parabolic Kepler motion, Eq. (3) with $m=1$.

Figure 1 visualizes that this solution has a natural continuation from the past history $t<0$ of the collision to the future $t>0$ through the singularity. The continuation corresponds to the analytic continuation of $r(t)$ onto a different sheet of the Riemann surface of the corresponding analytic function. The singularity at $t=0$ is an algebraic branch point of order 3.

In fact, a well developed theory of regularization in celestial mechanics has been around for almost a century. Whereas the beginnings may be attributed to Sundman (1907), the main ideas were contributed by Levi-Civita (1920). The notion *Levi-Civita regularization* is now used for the regularization of the binary collisions in the planar (2-dimensional) Kepler problem. The generalization to 3 dimensions was done by Kustaanheimo (1964) and Kustaanheimo-Stiefel (1965), their transformation often being abbreviated as *KS regularization*.

These regularization theories use a set of transformed variables such that the singularities due to binary collisions disappear from the motion as well as from the *differential equations* defining the motion. An important consequence of this property is the continuous dependence of the entire orbit upon initial conditions, even in the presence of binary collisions. An additional and important benefit of Levi-Civita's approach is that the regularized equations of motion of the unperturbed Kepler problem (e.g. the regularized version of Eq. (2)), are *linear*. This allows the development of simple perturbation theories.

In the following section we develop and summarize the principal aspects of Levi-Civita regularization. In Section 3 a theory of Kepler motion based on this regularization is proposed, whereas Section 4 presents the extension to three dimensions by means of quaternions. In Section 5 the Hamiltonian formalism in complex notation will be reviewed; then simultaneous regularization of several types of binary collisions by a single transformation will be discussed for two particular cases of the three-body problem. The final Section 6 will give an outlook to collisions of more than two bodies and to a few other topics not covered in detail here.

2. Levi-Civita Regularization

The fundamental object for discussing Levi-Civita regularization is the two-body problem or *Kepler problem*, i.e. the motion of a satellite of mass m_2 under the Newtonian force of a central body of mass m_1. By using relative coordinates with respect to m_1 there follows that the motion of m_2 with respect to m_1 is equivalent to the motion of a massless particle under the influence of a central body of mass $m = m_1 + m_2$. Since the motion takes place in a fixed plane we will use a two-dimensional coordinate system with the central body m at the origin.

Let $\mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$ be the position vector of the satellite with respect to the central body. For our purposes it will often be convenient to use complex notation $\mathbf{x} = x_1 + ix_2, \bar{\mathbf{x}} = x_1 - ix_2 \in \mathbb{C}$. In Sections 2 and 3 , bold face characters denote 2-vectors in complex notation, i.e., complex numbers. The bar means conjugation. The basic equation of motion is

$$\ddot{\mathbf{x}} = \mu \frac{\mathbf{x}}{r^3} = \mathbf{f}(\mathbf{x}, t), \qquad \mu = \gamma(m_1 + m_2), \qquad r = |\mathbf{x}|, \tag{4}$$

where r is the distance between the two bodies. For later use we have already introduced a perturbative force vector $\mathbf{f}(\mathbf{x}, t)$; the pure Kepler motion is obtained with $\mathbf{f} = \mathbf{0}$.

The corresponding energy equation is obtained by integrating the dot product $<,>$ of the vector $\dot{\mathbf{x}}$ with Eq. (4) (in vector version) with respect to t:

$$\frac{1}{2}|\dot{\mathbf{x}}|^2 - \frac{\mu}{r} = -h, \tag{5}$$

where h is the negative energy satisfying the differential equation and initial condition

$$\frac{dh}{dt} = -<\dot{\mathbf{x}}, \mathbf{f}(\mathbf{x}, t)>, \quad h(0) = \frac{\mu}{|\mathbf{x}(0)|} - \frac{1}{2}|\dot{\mathbf{x}}(0)|^2 \tag{6}$$

The sign of h has been chosen such that $h > 0$ corresponds to elliptic motion in the unperturbed Kepler problem.

The Levi-Civita regularization procedure will be carried out in three steps. The first step

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is the introduction of a new independent variable called *fictitious time* suggested by Sundman. The second step is the conformal squaring in the physical plane, which suggests using complex notation. In the final step the energy equation – also written in terms of the new variables – is used to eliminate the first derivatives. The formal calculations for determining the regularized equations of motion are very simple; they will be described in detail in the following three short subsections.

2.1. Time Transformation: Slow-Motion Movie

The first regularization step calls for introducing a fictitious time τ according to the Sundman transformation

$$dt = r \cdot d\tau, \qquad \frac{d}{d\tau}(\,) = (\,)'. \tag{7}$$

Therefore, the ratio $dt/d\tau$ of the two infinitesimal increments is made proportional to the distance r; the movie is run in slow-motion whenever r is small. Eqs. (4), (5) are transformed into

$$r\mathbf{x}'' - r'\mathbf{x}' + \mu\mathbf{x} = r^3\mathbf{f}, \qquad \frac{1}{2r^2}|\mathbf{x}'|^2 - \frac{\mu}{r} = -h. \tag{8}$$

2.2. Conformal Squaring

The second step of Levi-Civita's regularization procedure consists of representing the complex physical coordinate $\mathbf{x}$ as the square $\mathbf{u}^2$ of a complex variable $\mathbf{u} = u_1 + i\,u_2 \in \mathbb{C}$,

$$\mathbf{x} = \mathbf{u}^2, \tag{9}$$

i.e., the mapping from the parametric plane to the physical plane is chosen as a conformal squaring. As a consequence, the parametric $\mathbf{u}$-manifold is a Riemann surface with two sheets, connected by branch points at $\mathbf{u} = \mathbf{0}$ and at $\mathbf{u} = \infty$. Eq. (9) implies

$$r = |\mathbf{x}| = |\mathbf{u}|^2 = \mathbf{u}\bar{\mathbf{u}}, \tag{10}$$

and differentiation of Eqs. (9) and (10) with respect to τ yields

$$\mathbf{x}' = 2\mathbf{u}\,\mathbf{u}', \qquad \mathbf{x}'' = 2\left(\mathbf{u}\,\mathbf{u}'' + \mathbf{u}'^2\right) \in \mathbb{C}, \qquad r' = \mathbf{u}'\,\bar{\mathbf{u}} + \mathbf{u}\,\bar{\mathbf{u}}' \tag{11}$$

By substituting this into (8) we obtain

$$2\,r\,\mathbf{u}\,\mathbf{u}'' + \mathbf{u}^2\left(\mu - 2|\mathbf{u}'|^2\right) = r^3\mathbf{f}, \qquad \mu - 2|\mathbf{u}'|^2 = r\,h \tag{12}$$

where in the first expression the two terms $2\,r\,\mathbf{u}'^2 = 2\mathbf{u}'\,\bar{\mathbf{u}}\,\mathbf{u}\,\mathbf{u}'$ have cancelled out, and the second equation has been multiplied by $-r$.

Inverse map. Obtaining initial values $\mathbf{u}(0)=\sqrt{\mathbf{x}(0)}$ or $\mathbf{u}(0)=-\sqrt{\mathbf{x}(0)}$ requires the computation of a complex square root. This can conveniently be accomplished by means of the formula

$$\sqrt{\mathbf{x}}=\frac{\mathbf{x}+|\mathbf{x}|}{\sqrt{2\left(|\mathbf{x}|+\operatorname{Re}\mathbf{x}\right)}}, \tag{13}$$

which reflects the observation that the complex vector $\sqrt{\mathbf{x}}$ has the direction of the bisector between $\mathbf{x}$ and the real vector $|\mathbf{x}|$, and is therefore a vector proportional to $\mathbf{x}+|\mathbf{x}|$. Equation (13) holds in the range $-\pi<\arg(\mathbf{x})<\pi$ and represents the principal branch of the complex square root. The alternate formula

$$\sqrt{\mathbf{x}}=\frac{\mathbf{x}-|\mathbf{x}|}{i\sqrt{2\left(|\mathbf{x}|-\operatorname{Re}\mathbf{x}\right)}},$$

holds in $0<\arg(\mathbf{x})<2\pi$ and agrees with (13) in the upper half-plane.

2.3. Elimination of First Derivatives

The third step of Levi-Civita regularization produces linear differential equations for the unperturbed problem $\mathbf{f}=\mathbf{0}$ by replacing the parenthesis of Eq. (12_1) with rh from the energy relation (12_2) , and dividing by $r\,\mathbf{u}$, using (10). The result is

Theorem 1: The perturbed Kepler problem (4) with the energy equation (5) is equivalent with

$$2\mathbf{u}''+h\cdot\mathbf{u}=r\mathbf{f}\left(\mathbf{x},\mathbf{t}\right)\bar{\mathbf{u}} \text{ where } \mathbf{x}=\mathbf{u}^2\in\mathbb{C}, \qquad r=\mathbf{u}\bar{\mathbf{u}}$$

$$t'=r \tag{14}$$

$$h'=-\left\langle\mathbf{x}', f\left(\mathbf{x},t\right)\right\rangle, \qquad h(0)=\frac{\mu}{|\mathbf{x}(0)|}-\frac{1}{2}|\dot{\mathbf{x}}(0)|^2,$$

a system of differential equations for the dependent variables $\mathbf{u}\in\mathbb{C}$, $t\in\mathbb{R}$, $h\in\mathbb{R}$ as functions of fictitious time τ.

Regularization has been accomplished: all collisions, i.e. all passages of $\mathbf{u}$ through 0, do not cause singularities in the solution as long as the perturbation is regular. The following cases are of particular interest:

1. $f=0 \Rightarrow h=h(0)=\text{const}$. Equation (14) describes a harmonic (linear) oscillator in two dimensions, corresponding to unperturbed Kepler motion.
2. f has a potential $V(x)$, $f=-\text{grad}V \Rightarrow h(x)=h(0)+V(x)-V(0)$. Equation (14) describes a perturbed harmonic oscillator with varying frequency.
3. $f=O(\varepsilon)$, $\quad \varepsilon \to 0 \Rightarrow h(x)=h(0)+O(\varepsilon)$. Equation (14) describes a perturbed harmonic oscillator with slowly varying frequency.

The linear structure of the unperturbed version $\mathbf{f}=\mathbf{0}$ of Eq. (14) is a most welcome property for developing a simple theory of small perturbations. The basis is the explicit solution of the unperturbed problem, i.e. the linear harmonic oscillator $(14)_1$ with 4 integration constants, referred to as *orbital elements*. Owing to the linearity the perturbations of every order are determined by linear differential equations with only the inhomogeneity changing. For details see, e.g., Stiefel-Scheifele (1971), Waldvogel (2006). In contrast, classical perturbation theories based on the nonlinear equation (4) (see, *Classical Hamiltonian Perturbation Theory*) lead to more complicated equations for the higher-order perturbations.

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Biographical Sketch

Jörg Waldvogel, born 1938 in Zürich, Switzerland, received his Master’s degree (Diploma in Mathematics) 1962 at ETH Zurich (Swiss Federal Institute of Technology) and his PhD in 1966 at the same school under the supervision of Eduard Stiefel (Ref. [29], [30]). 1967 - 1970 he worked as a Research Scientist for Lockheed Missiles and Space Company, Huntsville, Alabama. 1968 – 1969 he was a part-time Assistant Professor in mathematics at the University of Alabama at Huntsville, and 1970 – 1972 an Assistant Professor at the University of Texas at Austin. As of 1972 he was the head of the Numerical Advisory Section at the Seminar for Applied Mathematics at ETH. 1985 he was given the title of a Professor, and he was teaching in various fields of pure and applied mathematics: differential equations, celestial mechanics, numerical mathematics, analysis, linear algebra. The emphasis of the past and current research is in the application of mathematics, mainly differential equations, to engineering and, in particular, to various topics of dynamical astronomy. After his retirement in 2003 he remained active in his fields of research as well as in teaching and in commissions at ETH.